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THREE-DIMENSIONAL CONTACT PROBLEMS TAKING FRICTION AND NON-LINEAR ROUGHNESS INTO ACCOUNT†

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Three-dimensional contact problems for an elastic layer with surface roughness are considered taking into account Coulomb friction in the previously unknown area of contact. It is assumed that the deformation of the microprotuberances of the elastic surface in contact with a rigid punch is non-linear (for example, it obeys a power law) [1, 2]. The solution of the problems is reduced to investigating non-linear Hammerstein integral equations, for which the existence and uniqueness of the solution is proved, and also the applicability of the method of successive approximations. Numerical results are presented which show the effect of the roughness and the friction force on the contact pressure, the dimensions of the contact area, and the relation between the indenting force and the indentation of the punch. © 2004 Elsevier Ltd. All rights reserved.

The plane contact problem for a rough elastic solid was apparently investigated for the first time in [3] using a linear law of deformation of microprotuberances. However, a number of experimental investigations [4, 5] has shown that there is a power-law relation between the closeness of the contacting rough surfaces and the pressure. Using this relation investigations have been made [1, 2] of the plane and axisymmetric contact problems for the known contact area and ignoring friction forces, and references have been made to a number of previous investigations. The method of solving contact problems with an unknown contact area, used in this paper, was proposed in [6] for analysing the three-dimensional problem for a rough elastic half-space for a general non-linear law of deformation of the microprotuberances and ignoring friction forces. When proving the existence and uniqueness of the solution, the monotonicity of a certain non-linear integral operator was used in [6] as well as the conjugacy of the linear integral operator generated by Green's function for a half-space. The corresponding proofs, given below, are not based on these properties, since the friction forces lead to non-self-conjugacy, and monotonicity breaks down for cases when the characteristic dimensions of the contact area are of the order of the layer thickness. Three-dimensional contact problems ignoring the roughness in quasi-static formulation in which the friction forces are taken into account as in this paper, only in the direction of motion of the punch, were considered previously in [7–9]. The method proposed in [6] is used in [8–10] to determine the unknown contact areas of elastic bodies of different shapes. Attempts have been made to investigate the contact of rough surfaces based on statistical and fractal approaches to describe the unknown distribution of the microprotuberances [11].

1. FORMULATION AND REDUCTION TO A NON-LINEAR INTEGRAL EQUATION

Consider an infinite layer $\{-\infty < x, y < \infty,\ 0 \leqslant z \leqslant h\}$ with elastic parameters G (the shear modulus) and ν (Poisson's ratio). We will investigate the quasi-static contact problem of a rigid punch, which is initially indented into the face $z = h$ of the layer, and then begins to move slowly over this face (without sag) along the x axis (see Fig. 1). The lower face $z = 0$ of the layer is fixed (problem A) or is in conditions of sliding clamping (problem B). The punch has a smooth base, described by the function $f(x, y)$, such that the unknown contact region Ω is elongated along the y axis (in the sense of the ratio of the parallel x and y axes of the sides of the minimum rectangle containing Ω). We can then neglect the Coulomb friction forces in the direction of the y axis and only take into account the friction forces collinear with the direction of motion. Because of the smoothness of the base of the punch, the contact pressure should vanish on the boundary of the contact area $\partial\overline{\Omega}$. The punch is acted upon by an normal force P, applied at a distance $|d|$ from the z axis, and a shear force $T = \mu P$, applied at a height e above the face $z = 0$

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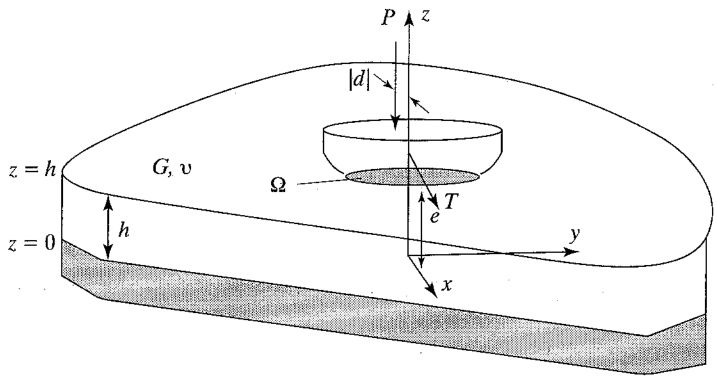

Fig. 1

of the layer, μ is the Coulomb friction coefficient. If $d < 0$, the force P is applied along the negative x semiaxis. The problems are symmetrical about the y coordinate. When $\mu > 0$ the punch moves in the positive direction of the x axis, and when $\mu < 0$ it moves in the opposite direction. The unknown normal contact pressure $\sigma_z = -q(x, y)$, $(x, y) \in \Omega$ is related to the shear stress by Coulomb's law $\tau_{xz} = -\mu q(x, y)$. The surface of the layer has microroughnesses as a result of processing, which, on contact with the punch, make an additional contribution u_z^a to the normal displacement of the layer boundary, which is defined as follows:

$$u_z^a = -\Phi[q(x, y)] \quad (x, y) \in \Omega \tag{1.1}$$

where $\Phi(t)$ $(-\infty < t < \infty)$ is a continuous (non-linear) function, strictly increasing when $t > 0$ and equal to zero when $t \leqslant 0$. When $t > 0$ an inverse function $H[\Phi(t)] = t$ exists for the function $\Phi(t)$.

This model for taking into account the roughness of the surface is widely used at the present time [1–6]. The model is employed when the density of the actual contact areas are fairly large. In applications, the function $\Phi(t)$ is often approximated by a power function, which will not be done below. Note, in addition, that the height and density of the microroughnesses of the surface also affect the choice of the value of the friction coefficient μ. This question remains open here and will be investigated later.

The condition for contact between the bodies can be written in the form

$$z = h: u_z + u_z^a = -[\delta - f(x, y)] \; (x, y) \in \Omega \tag{1.2}$$

where δ is the indentation of the punch. The normal displacement $u_z(x, y, z)$ must satisfy the three-dimensional Lamé equations of equilibrium and the boundary conditions

$$\begin{aligned}
&z = h: \sigma_z = -q(x, y), \quad \tau_{xy} = \mu\sigma_z, \quad \tau_{yz} = 0, \quad (x, y) \in \Omega \\
&\sigma_z = \tau_{xz} = \tau_{yz} = 0, \quad (x, y) \notin \Omega \\
&z = 0: u_x = u_y = u_z = 0 \quad \text{(Problem A)} \\
&z = 0: u_z = \tau_{xz} = \tau_{yz} = 0 \quad \text{(Problem B)}
\end{aligned} \tag{1.3}$$

In addition, the stresses must disappear at infinity.

The formulation of the contact problems is as follows. For specified functions $f(x, y)$ and $\Phi(t)$ and the quantities G, ν, μ and δ it is required to determine the contact area Ω, the contact pressure $q(x, y)$, the external forces P and T, and also their branches d and e. It is also possible to specify the force P, knowing the unknown indentation δ.

Solving the boundary-value problems (1.3) using a double Fourier transformation, we obtain, in particular, the representation

$$u_z(x, y, h) = -\frac{1}{2\pi\theta}\int_\Omega K(x-\xi, y-\eta) q(\xi, \eta) d\Omega_{\xi\eta}, \quad \theta = \frac{G}{1-\nu} \tag{1.4}$$

where

$$K(t,\tau) = K_1(t,\tau) - \epsilon K_2(t,\tau), \quad \epsilon = \mu\frac{1-2\nu}{2-2\nu}$$

$$K_n(t,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L_n^*(\alpha,\beta)\exp[-i(\alpha t+\beta\tau)]d\alpha d\beta, \quad n = 1,2 \tag{1.5}$$

$$L_1^*(\alpha,\beta) = L_1(\gamma)/\gamma, \quad L_2^*(\alpha,\beta) = i\alpha L_2(\gamma)/\gamma^2, \quad \gamma = \sqrt{\alpha^2+\beta^2}$$

The functions $L_n(u)$ $(n = 1, 2)$ have the form

$$\begin{gathered} L_1(u) = [2\kappa\,\mathrm{sh}(2uh) - 4uh]/L_0(u), \; L_2(u) = [2\kappa\,\mathrm{ch}(2uh) - 4(1-2\nu)^{-1}u^2h^2 - 2\kappa]/L_0(u) \\ L_0(u) = 2\kappa\,\mathrm{ch}(2uh) + 4u^2h^2 + 1 + \kappa^2, \quad \kappa = 3-4\nu \end{gathered} \tag{1.6}$$

for problem A, and

$$\begin{gathered} L_1(u) = [\mathrm{ch}(2uh) - 1]/L_0(u), \quad L_2(u) = [\mathrm{sh}(2uh) - 2(1-2\nu)^{-1}uh]/L_0(u) \\ L_0(u) = \mathrm{sh}(2uh) + 2uh \end{gathered} \tag{1.7}$$

for problem B.

Using integral 8.441.2 of [12]

$$\int_0^{\pi/2} \cos(a\cos\chi)\cos(b\sin\chi)d\chi = \frac{\pi}{2}\mathrm{J}_0(\sqrt{a^2+b^2}) \tag{1.8}$$

and the relation $J_0'(z) = -J_1(z)$, where $J_n(u)$ $(n = 0, 1)$ is the Bessel function, we reduce expression (1.5) to the form

$$\begin{gathered} K_1(x-\xi, y-\eta) = \int_0^{\infty} L_1(u)\mathrm{J}_0(uR)du \\ K_2(x-\xi, y-\eta) = \frac{x-\xi}{R}\int_0^{\infty} L_2(u)\mathrm{J}_1(uR)du \\ R = \sqrt{(x-\xi)^2+(y-\eta)^2} \end{gathered} \tag{1.9}$$

Taking into account the asymptotic behaviour: $L_n(u) \to 1$ as $u \to +\infty$ $(n = 1, 2)$, and also integral 6.511.1 from [12]

$$\int_0^{\infty} \mathrm{J}_n(uR)du = \frac{1}{R}, \quad n = 0, 1 \tag{1.10}$$

we isolate the principal terms in the kernels (1.9)

$$\begin{gathered} K_1(x-\xi, y-\eta) = \frac{1}{R} + \int_0^{\infty}[L_1(u) - 1]\mathrm{J}_0(uR)du \\ K_2(x-\xi, y-\eta) = \frac{x-\xi}{R^2} + \frac{x-\xi}{R}\int_0^{\infty}[L_2(u)-1]\mathrm{J}_1(uR)du \end{gathered} \tag{1.11}$$

Now substituting representations (1.1) and (1.4) into contact condition (1.2), we obtain the following non-linear integral equation in the contact pressure $q(x, y)$

$$\Phi[q(x, y)] + \frac{1}{2\pi\theta}\int_{\Omega} K(x-\xi, y-\eta)q(\xi, \eta)d\Omega_{\xi\eta} = \delta - f(x, y), \quad (x, y) \in \Omega \tag{1.12}$$

2. THE SOLUTION OF EQ. (1.12)

We will assume that the unknown contact area Ω is contained inside the rectangle $S = \{(x,y) : |x| \leqslant a, |y| \leqslant b\}$ $(b \geqslant a)$. The contact pressure must be positive inside Ω and equal to zero in the additional region $S\backslash\Omega$. Suppose Ω is an open set. Since the region $S\backslash\bar{\Omega}$ is contact-free, the non-penetration condition $u_z(x, y, h) > \delta - f(x, y)$ is satisfied in it. Combining all these conditions and extending the integration in formula (1.12) to the rectangle S, we reduce the contact problems to the following relations

$$\begin{aligned} &\Phi[q(M)] + \frac{1}{2\pi\theta}\int_S K_*(M, N)q(N)dS_N = g(M) \wedge q(M) > 0, \quad M \in \Omega \\ &\frac{1}{2\pi\theta}\int_S K(M, N)q(N)dS_N > g(M) \wedge q(M) = 0, \quad M \in (S\backslash\bar{\Omega}) \end{aligned} \tag{2.1}$$

where we have used the notation

$$M = (x, y), \quad N = (\xi, \eta), \quad g(M) = \delta - f(M), \quad K_*(M, N) = K(x-\xi, y-\eta) \tag{2.2}$$

After determining the function $q(x,y)$ and the region Ω from system (2.1), we can obtain the quantities P, d and e from the following three conditions of equilibrium of the punch (see formula (14) in [7])

$$\int_{\Omega} q(M)d\Omega_M = P, \quad \int_{\Omega} xq(M)d\Omega_M = Pd, \quad e - h = -\frac{d}{\mu} \tag{2.3}$$

System (2.1) can be reduced to a single non-linear equation in the rectangle S. The region in which the solution of this equation is positive will be the contact region [6].

We will assume that a bounded region $S_0 = \{M : g(M) > 0\}$ exists such that $\Omega \subset \bar{S}_0 \subset S$. We will introduce the notation

$$w(M) = \Phi[q(M)], \quad q(M) = H[w(M)], \quad \lambda_* = (2\pi\theta)^{-1} \tag{2.4}$$

and the (non-linear) operators

$$\mathcal{H}v(M) := \begin{cases} H[v(M)], & v(M) > 0 \\ 0, & v(M) \le 0 \end{cases}, \quad \mathcal{Q}v(M) := \begin{cases} v(M), & v(M) > 0 \\ 0, & v(M) \le 0 \end{cases} \tag{2.5}$$

We will rewrite system (2.1) using formulae (2.4) and (2.5) in the equivalent form

$$\begin{aligned} &w(M) + \lambda_*\int_S K_*(M, N)\mathcal{H}w(N)dS_N = g(M) \wedge w(M) > 0, \quad M \in \Omega \\ &\lambda_*\int_S K_*(M, N)\mathcal{H}w(N)dS_N > g(M) \wedge w(M) = 0, \quad M \in (S\backslash\bar{\Omega}) \end{aligned} \tag{2.6}$$

Consider the Hammerstein integral equation

$$v(M) + \lambda_*\int_S K_*(M, N)\mathcal{H}v(N)dS_N = g(M), \quad M \in \Omega \tag{2.7}$$

Equation (2.7) can also be written in operator form

$$v + \lambda_*\mathcal{K}\mathcal{H}v = g \tag{2.8}$$

where $\upsilon = \upsilon(M)$, $g = g(M)$ and $\mathcal{K}$ is an integral operator of the form

$$\mathcal{K}\upsilon := \int_S K_*(M, N)\upsilon(N)dS_N \tag{2.9}$$

We will further assume that $g(M) \in \mathcal{C}(S)$, where $\mathcal{C}(S)$ is a Banach space of functions continuous in the rectangle S. We will seek solutions of system (2.1) and Eq. (2.8) in the same space.

Theorem 1. If $\upsilon_* = \upsilon_*(M) \in \mathcal{C}(S)$ is a solution of Eq. (2.8), then $w = \mathfrak{Q}\upsilon_*$, $\Omega = \{M\colon \upsilon_*(M) > 0\}$ is a solution of system (2.6), and $\Omega \neq \emptyset$ when $S_0 \neq \emptyset$; conversely, if $w = w(M) \in \mathcal{C}(S)$ satisfies system (2.6), then

$$\upsilon_* = g - \lambda_*\mathcal{K}\mathcal{H}w, \quad M \in S \tag{2.10}$$

is a solution of Eq. (2.8).

Proof. We will first show that $\Omega \neq \emptyset$ if $S_0 \neq \emptyset$. Assume the opposite. Then the inequality $g \leqslant 0$ follows from Eq. (2.8), which contradicts the existence of $S_0 \neq \emptyset$. We have used the first definition of (2.5) here.

Suppose υ_* is the solution of Eq. (2.8). Note that $\mathcal{H}\upsilon_* = \mathcal{H}w$, where $w = \mathfrak{Q}\upsilon_*$ (see formula (2.5)). When $M \in \Omega$ we have $w = \upsilon_* > 0$ and $w + \lambda_*\mathcal{K}\mathcal{H}w = g$. If $M \in (S\backslash\overline{\Omega})$, then $\upsilon_* = g - \lambda_*\mathcal{K}\mathcal{H}\upsilon_* < 0$, $\lambda_*\mathcal{K}\mathcal{H}w > g$, $w = 0$ and this proves the direct assertion of the theorem.

Now suppose $w = w(M)$ is a solution of system (2.6). For $M \in \Omega$ the equality $\upsilon_* = w$ follows from the first equation of (2.6) and from relation (2.10). If $M \in (S\backslash\overline{\Omega})$, we have $\upsilon_* = g - \lambda_*\mathcal{K}\mathcal{H}w < 0$, whence we obtain $\mathcal{H}\upsilon_* = \mathcal{H}w$ in the rectangle S. We can now rewrite formula (2.10) in the form

$$\upsilon_* = g - \lambda_*\mathcal{K}\mathcal{H}\upsilon_*, \quad M \in S \tag{2.11}$$

i.e. υ_* is a solution of Eq. (2.8).

Lemma 1. Being non-selfconjugate, the integral operator $\mathcal{K}$ (2.9) is strictly positive in $\mathcal{L}_2(S)$, i.e.

$$(\mathcal{K}q, q)_{\mathcal{L}_2(S)} > 0 \tag{2.12}$$

where $q \neq 0$.

Proof. From formulae (1.5) we have

$$\begin{aligned} K(t, \tau) &= \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{L_1(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}}\cos(\alpha t)\cos(\beta\tau)d\alpha d\beta - \\ &- \epsilon\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\alpha L_2(\sqrt{\alpha^2+\beta^2})}{\alpha^2+\beta^2}\sin(\alpha t)\cos(\beta\tau)d\alpha d\beta \end{aligned} \tag{2.13}$$

We will put

$$\begin{aligned} &\int_S q(x, y)\cos(\alpha x)\cos(\beta y)dS_{xy} = C_c(\alpha, \beta) \\ &\int_S q(x, y)\sin(\alpha x)\cos(\beta y)dS_{xy} = C_s(\alpha, \beta) \\ &\int_S q(x, y)\cos(\alpha x)\sin(\beta y)dS_{xy} = S_c(\alpha, \beta) \\ &\int_S q(x, y)\sin(\alpha x)\sin(\beta y)dS_{xy} = S_s(\alpha, \beta) \end{aligned} \tag{2.14}$$

Then, taking into account the fact that

$$\begin{aligned} &\int_0^\infty \frac{\alpha L_2(\sqrt{\alpha^2+\beta^2})}{\alpha^2+\beta^2}[C_s(\alpha, \beta)C_c(\alpha, \beta) - C_c(\alpha, \beta)C_s(\alpha, \beta) + \\ &+ S_s(\alpha, \beta)S_c(\alpha, \beta) - S_c(\alpha, \beta)S_s(\alpha, \beta)]d\alpha d\beta = 0 \end{aligned}$$

we obtain

$$(\mathcal{K}q, q)_{\mathcal{L}_2(S)} = \\ = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{L_1(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}}[C_c^2(\alpha,\beta)+C_s^2(\alpha,\beta)+S_c^2(\alpha,\beta)+S_s^2(\alpha,\beta)]d\alpha d\beta > 0 \tag{2.15}$$

with $q(x, y) \not\equiv 0$, since the function $L_1(u)$ is positive when $u \in (0, \infty)$ and $\nu \in [0, 1/2]$ for both contact problems A and B.

Theorem 2. If Eq. (2.8) has a solution in $\mathcal{C}(S)$, it is unique.

Proof. Suppose we have two solutions: υ_1 and υ_2 where $\upsilon_1 \neq \upsilon_2$. Subtracting Eq. (2.8) with $\upsilon = \upsilon_1$ from the same equation with $\upsilon = \upsilon_2$, we obtain

$$d_0 + \lambda_* \mathcal{K}d = 0; \quad d_0 = \upsilon_2 - \upsilon_1, \quad d = \mathcal{H}\upsilon_2 - \mathcal{H}\upsilon_1 \tag{2.16}$$

We multiply Eq. (2.16) by d and integrate the result over the rectangle S. We obtain the relation

$$(d_0, d)_{\mathcal{L}_2(S)} + (\lambda_* \mathcal{K}d, d)_{\mathcal{L}_2(S)} = 0$$

the second term in which is positive when $d \neq 0$, as result of Lemma 1, while the first term is also positive, since the inequality $\mathcal{H}\upsilon_2(M) \geqslant \mathcal{H}\upsilon_1(M)$ follows from the inequality $\upsilon_2(M) > \upsilon_1(M)$, since $\mathcal{H}$ is a monotonic operator. Consequently, $d = 0$, whence we have $d_0 = 0$ (see Eq. (2.16)).

Equation (2.8) can also be written in the form

$$\upsilon = \mathcal{U}\upsilon; \quad \mathcal{U}\upsilon := g - \lambda_* \mathcal{K}\mathcal{H}\upsilon \tag{2.17}$$

In order to prove that a solution of Eq. (2.8) or (2.17) exists using Shauder's principle [13], it is sufficient to show that the operator $\mathcal{U}$, completely continuous in $\mathcal{C}(S)$, transfers a certain convex set of the Banach space $\mathcal{C}(S)$ into a compact subset of this set. This has been shown (see [6], Note 5) for the case of the contact problem for a half-space ($h \to +\infty$), ignoring friction forces and using the monotonicity of the operator $\mathcal{K}$ (2.9), guaranteed by the positiveness of its kernel $K_* = 1/R$ (2.2) in the rectangle S. However, the positivity of K_* and the monotonicity of $\mathcal{K}$, as the following two examples show, in the case of a layer is only preserved for fairly large values of h/b for fairly small values of μ. Note that the friction coefficient μ for metal surfaces usually satisfies the condition $|\mu| \leqslant 0.2$.

Example 1. Consider the axisymmetric problems A_0 and B_0 for a layer loaded with a concentrated force p at the point $x = y = 0, z = h$. The boundary conditions at $z = 0$ are the same as for problems A and B respectively (see formulae (1.3)). For the dimensionless normal displacement

$$u(\rho) = -u_z(x, y, h)\frac{2\pi\theta h}{p}, \quad \rho = \frac{r}{h}, \quad r = \sqrt{x^2+y^2}$$

using formulae (1.4) and (.1.5) with $\epsilon = 0$ and the first formula of (1.11), we obtain the expression

$$u(\rho) = \frac{1}{\rho} + \int_0^\infty [L(t) - 1]J_0(t\rho)dt, \quad L(t) = L_1\left(\frac{u}{h}\right) \tag{2.18}$$

The function $L_1(u)$ is defined by the first formula of (1.6) or (1.7). Clearly the kernel K_* for contact problems A and B (without friction) will be positive for those values of R for which the corresponding function $u(R/h)$ is positive. The first term $1/\rho$ in formula (2.18) is the displacement for an elastic half-space. The functions $u(\rho)$ for certain ρ with $\nu = 0.3$ are presented below:

ρ	0.5	1	1.24	1.52	2	
$u(\rho)$	0.765	0.0583	0	–0.0161	–0.0107	(Problem A_0)
$u(\rho)$	0.924	0.129	0.0388	0	–0.0106	(Problem B_0)

It can be seen that the kernel K_* in Eq. (2.8) for both problems with $\mu = 0$ and $\nu = 0.3$ will be negative, for example, when $R = 2h$. The surface of the layer is deformed both in the positive and the negative directions of the z axis. This effect is due to the influence of the bottom $z = 0$ of the layer of finite thickness.

Example 2. For the limiting case $h \to \infty$, corresponding to an elastic half-space, the kernel K_* (2.2) takes the form

$$K_*(M, N) = \frac{1}{R} - \epsilon\frac{x-\xi}{R^2} \tag{2.19}$$

The function K_* (2.19) is positive in the rectangle S if $|\epsilon| < 1$. When $\nu = 0.3$ the latter condition takes the form $|\mu| < 3.5$.

Lemma 2. Suppose the kernel K_* (M, N) is positive when $M, N \in S$. Then the operator (2.17) maps the set $B = \{\chi: \mathcal{U}g \leq \chi \leq g\} \subset \mathcal{C}(S)$ into itself, and a solution $\upsilon_* \in B$ of Eq. (2.17) or (2.8) exists.

Proof. It is clear that $\mathcal{U}g < g$ in S $(g \neq 0)$. Suppose $\mathcal{U}g \leq \chi \leq g$. Then, at any point M

$$\chi = t\mathcal{U}g + (1-t)g, \quad t = t(M) \in [0; 1]$$

whence

$$\chi = g - t\lambda_*\mathcal{K}\mathcal{H}g, \quad \mathcal{U}\chi = g - \lambda_*\mathcal{K}\mathcal{H}(g - t\lambda_*\mathcal{K}\mathcal{H}g)$$

The double inequality

$$\mathcal{U}g \leq \mathcal{U}\chi \leq g \tag{2.20}$$

can be rewritten in the form of an inequality

$$\lambda_*\mathcal{K}\mathcal{H}g \geq \lambda_*\mathcal{K}\mathcal{H}(g - t\lambda_*\mathcal{K}\mathcal{H}g) \geq 0$$

which is satisfied in view of the monotonicity of the operator $\mathcal{H}$ and $\mathcal{K}$, since $K_* > 0$. Consequently, the double inequality (2.20) is also satisfied. Equation (2.17) or (2.8) has a solution by virtue of Shauder's principle.

For both contact problems A and B the condition $K_* > 0$ in the region S is satisfied, for example, when $h/b > 2.3$ and $\mu = 0$. If the function K_* changes sign in S, the existence of a solution of Eq. (2.17) can be established using the following lemma.

Lemma 3. Suppose the functions belong g to the open sphere $B_\rho \subset \mathcal{C}(S)$ of radius ρ with centre $\|g\| < \rho$. Suppose T is the boundary of $\overline{B}_\rho$. Then $\mathcal{U}T \subset \overline{B}_\rho$ for fairly small values of λ_*, and Eq. (2.17) or (2.8) has the solution $\upsilon_* \in B_\rho$.

To prove this it is sufficient to use another formulation of Shauder's principle [14] and the fact that $\mathcal{U}$ is a completely continuous operator.

Further, for the displacement of the roughnesses we will use the power law [2]

$$\Phi[q(M)] = Aq^\beta(M) \quad (0 < \beta \leq 1)$$

and we replace the last formula of (2.4) by

$$\lambda_* = (2\pi\theta A^\gamma)^{-1} \quad (\gamma = 1/\beta \geq 1) \tag{2.21}$$

Then, we must take as the function $H[\upsilon(M)]$ in the first formula of (2.5)

$$H[\upsilon(M)] = \upsilon^\gamma(M) \tag{2.22}$$

Smallness of the parameter λ_* in formulating Lemma 3 can be achieved for fairly large values of the parameter A, characterizing the roughness with which the elastic surface of the layer is processed.

Equation (2.17) will be solved using successive approximations. We will investigate the convergence of this process using the Lipschitz condition for the non-linear operator [15], by estimating the constant in the Lipschitz condition using the Frechet derivative.

Lemma 4. The non-linear operator $\mathcal{U}$ in any closed sphere $B_\rho \subset \mathcal{C}(S)$ of radius ρ satisfies the Lipschitz condition with constant $q_0 = \lambda_*\|\mathcal{K}\|\gamma\rho^{\gamma-1}$.

Proof. We can regard the function H (2.22) as a non-linear operator, defined on non-negative functions with Frechet derivative $H'(\upsilon)h = \gamma\upsilon^{\gamma-1}h$. It is obvious that $\mathcal{K} = H\mathcal{Q}$. The operator H (2.22) satisfies the Lipschitz condition on a set of non-negative bounded functions $\chi \in \mathcal{C}(S)$: $0 \leq \chi \leq \rho\}$ with constant $q_1 = \gamma\rho^{\gamma-1}$. We further have

$$\|\mathcal{U}u - \mathcal{U}\upsilon\| = \lambda_*\|\mathcal{K}(\mathcal{H}u - \mathcal{H}\upsilon)\| \leq \lambda_*\|\mathcal{K}\|\|\mathcal{H}u - \mathcal{H}\upsilon\| \leq$$

$$\leq \lambda_*\|\mathcal{K}\|q_1\|\mathcal{Q}u - \mathcal{Q}\upsilon\| \leq \lambda_*\|\mathcal{K}\|q_1\|u - \upsilon\| = q_0\|u - \upsilon\| \tag{2.23}$$

for any $u, \upsilon \in B_\rho$.

Theorem 3. Suppose the operator $\mathcal{U}$ maps the closed sphere $B_\rho \subset \mathcal{C}(S)$ of radius ρ into itself and let us assume that $q_0 = \lambda_*\|\mathcal{K}\|\gamma\rho^{\gamma-1} < 1$. Then, for any initial element $\upsilon_0 \in B_\rho$, successive approximations

$$\upsilon_n = \mathcal{U}\upsilon_{n-1}, \quad n = 1, 2, \ldots \tag{2.24}$$

converge to a unique solution (by Theorem 2) of Eq. (2.17).

Proof. The operator $\mathcal{U}$ is compressive by Lemma 4. The rest repeats the proof of Theorems 1.1 and 1.2 of [15].

The constant q_0 in the Lipschitz condition depends on the parameter λ_* (2.21), λ and ρ, as well as on the norm of the linear operator $\mathcal{K}$. The smaller the norm $\|g\|$, the less the value of ρ that can be taken for the same large values of A.

Example 3. The norm $\|\mathcal{K}\|$ can easily be calculated for the limiting case of a half-space, $h/b \to \infty$, for which

$$\mathcal{K}\upsilon = \int_S \left(\frac{1}{R} - \epsilon\frac{x-\xi}{R^2}\right)\upsilon dS_{\xi\eta} \tag{2.25}$$

by formula (2.19). Suppose $|\epsilon| < 1$; then the kernel of the operator (2.25) is positive in the region S, and consequently

$$\|\mathcal{K}\| = \max_{(x,y)\in S}\int_S \left(\frac{1}{R} - \epsilon\frac{x-\xi}{R^2}\right)dS_{\xi\eta} \tag{2.26}$$

This integral is easily evaluated using formulae 1.6.8.14 and 1.6.7.3 of [16]

$$F(s) := -\int\frac{\ln(s+\sqrt{s^2+1})}{s^2}ds = \frac{\ln(s+\sqrt{s^2+1})}{s} + \ln\left|\frac{1+\sqrt{s^2+1}}{s^2}\right|$$
$$G(s,a) := \int\ln(s^2+a^2)ds = s\ln(s^2+a^2) - 2s + 2a\,\mathrm{arctg}\frac{s}{a} \tag{2.27}$$

Finally we obtain

$$\|\mathcal{K}\| = \max_{(x,y)\in S}(N_1 - \epsilon N_2) \tag{2.28}$$

where

$$N_1 = N_1^+ + N_1^-, \quad N_1^\pm = (b\pm y)\left[F\left(\frac{b\pm y}{a-x}\right) + F\left(\frac{b\pm y}{a+x}\right)\right]$$

$$N_2 = N_2^+ + N_2^-, \quad N_2^\pm = \pm\frac{1}{2}[G(b-y, a\pm x) + G(b+y, a\pm x)]$$

When there is no friction, $\mu = \epsilon = 0$, a maximum is reached in (2.28) when $x = y = 0$, giving $\|\mathcal{K}\| = 4bF(b/a)$. Hence, in particular, we obtain that $\|\mathcal{K}\| = 8b\ln(1+\sqrt{2}) \approx 7.05b$ when $a = b$ and $\|\mathcal{K}\| \approx 2.64b$ when $a = 0.2b$. When there is friction, $\epsilon \neq 0$, a maximum is reached in (2.28) at the point $(x_m, 0)$, and the value of x_m is found from the transcendental equation

$$\ln(x_- + \sqrt{x_-^2 + 1}) + \ln(x_+ + \sqrt{x_+^2 + 1}) + \epsilon(\operatorname{arctg} x_- - \operatorname{arctg} x_+) = 0$$

where we have put

$$x_- = \frac{b}{x-a}, \quad x_+ = \frac{b}{x+a}$$

In particular, when $a = b$, $\mu = 0.2$ and $\nu = 0.3$ we obtain $x_m \approx -0.06b$ and $\|\mathcal{K}\| \approx 7.06b$.

3. NUMERICAL ANALYSIS

Suppose the base of the punch has the form of an elliptical paraboloid

$$f(x, y) = x^2/(2R_1) + y^2/(2R_2), \quad R_1 \le R_2$$

We will introduce the following dimensionless quantities

$$x_* = \frac{x}{b}, \quad y_* = \frac{y}{b}, \quad \lambda = \frac{h}{b}, \quad \delta_* = \frac{\delta}{b}, \quad \varepsilon = \frac{a}{b}, \quad A_0 = \frac{b}{2R_1}, \quad B_0 = \frac{b}{2R_2}$$

$$A_* = \frac{A(2\pi\theta)^\beta}{b}, \quad P_* = \frac{P}{2\pi\theta b^2}, \quad q_*(x_*, y_*) = \frac{q(x, y)}{2\pi\theta} \tag{3.1}$$

$$g_*(x_*, y_*) = \delta_* - A_0 x_*^2 - B_0 y_*^2$$

etc. In the notation (3.1) the regions S and Ω are also changed. The asterisks will henceforth be omitted. The parameter λ represents the relative thickness of the layer. Instead of (2.17) we obtain a dimensionless non-linear equation of the form

$$v = g - A^{-2}\mathcal{K}\mathcal{H}v \tag{3.2}$$

Equation (3.2) was solved by the method of successive approximations, which converges for fairly large values of A (depending on $\|g\|$, $\|\mathcal{K}\|$ and γ).

The calculation were carried out for problem A with $\nu = 0.3$ and $\beta = 0.4$ (this value of β is borrowed from [2], where $A \sim 1$ is also taken). Values of P as a function of λ for different values of A (the smaller the value of A the better the surface of the layer is processed) for the case

$$\delta = 0.05, \quad A_0 = 0.2, \quad B_0 = 0.05, \quad \varepsilon = 0.5, \quad \forall\mu \in [0, 0.2] \tag{3.3}$$

are given below:

λ	2.5	2	1.5	1	0.5	0.25	
$P \times 10^3$	0.405	0.407	0.409	0.413	0.421	0.429	(for $A = 0.8$)
$P \times 10^3$	1.13	1.14	1.16	1.19	1.26	1.33	(for $A = 0.5$)

The fact that P is independent (to the first few significant digits) of the friction coefficient μ (over this range of values of μ) was also pointed out in [8] when investigating the contact problem for a half-space ignoring the roughness ($A = 0$).

When there is friction the contact pressure becomes asymmetrical, but its integral characteristic stays the same, which can be explained in the same way as before [8], by expanding the pressure in terms of the small parameter ε (see the second formula of (1.5)). The force is independent of the sign of μ, and hence the effective friction will be described by small terms of the order of ε^2. For a specified accuracy the number of iterations falls as λ decreases, which is due to the reduction in the norm $\|\mathcal{K}\|$. In Fig. 2(a) we show graphs of $P(\delta)$ for $A = 0.8$ (the continuous curve) and $A = 0.7$ (the dashed curve), where $\lambda = 1$, and the values of the remaining parameters are given by Eqs (3.3). For the same value of the force, the indentation is greater for a less ground surface, for which the value of A is greater. The contact area also increases as A increases, while the pressure at the centre of this region is reduced. This is shown in Fig. 2(b) for the case when

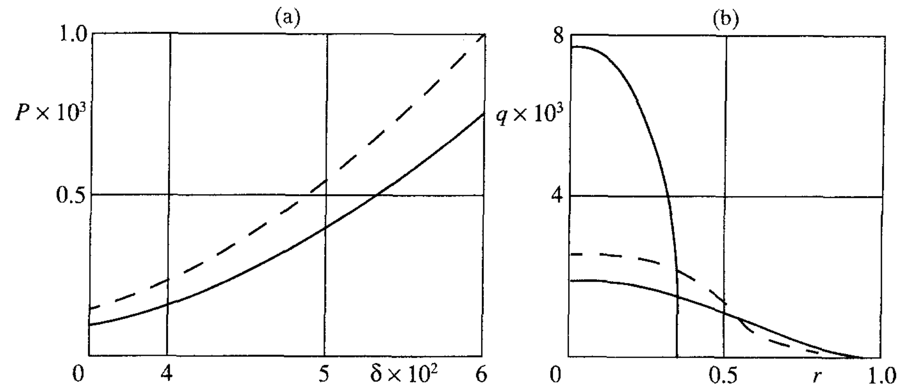

Fig. 2

$$\lambda \to \infty, \quad \mu = 0, \quad P = 0.00199, \quad A_0 = B_0 = 0.05$$

when, in view of the axial symmetry, the pressure depends only on the radial coordinate r (this refers to b). The thin line corresponds to the exact solution of the problem for $A = 0$ [10] ($\delta = 0.0130$ and the radius ρ of the region Ω is equal to 0.36), the dashed curve corresponds to $A = 0.3$ ($\delta = 0.0370$ and $\rho = 0.82$), while the thick curve corresponds to $A = 0.5$ ($\delta = 0.0500$ and $\rho = 0.96$).

The effect of the roughness, in the sense of the difference from the corresponding solution for $A = 0$, increases particularly for contact areas whose characteristic dimensions considerably exceed the value of the indentation of the punch.

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